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Inverse Rational L^1 Approximation

R. C. GAYLE

Department of Science and Mathematics, Montana State University-Northern, P.O. Box 7751, Havre, Montana 59501

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We consider the nonlinear approximating family R^n_m of rational expressions over a real interval. In the L^p norms, $1 < p < \infty$ non-normal elements of this family cannot arise as best approximations to functions outside the family. In the L^1 case, Dunham (1971) has shown that for a continuous function no rational of defect two or greater, excepting the rather special case of the function 0, can be a best approximation. Cheney and Goldstein have shown (1967) that any normal rational function can arise as the best approximation to some function $f \in L^2$ which is not in the rational family. We show here that there exist continuous functions not in R^n_m which do have any given defect one functions as their best approximations by using variational techniques from Wolfe (1976). © 1995 Academic Press, Inc.

In what follows we consider approximation questions involving the family $R^n_m[a, b]$ of rational functions over a given real interval $[a, b]$, the precise definition of which will follow. $R^n_m[a, b]$ is a nonlinear approximating family. We concern ourselves with the inverse existence question. That is, we ask which elements of $R^n_m[a, b]$ may arise as best approximations to continuous functions from outside the family? Recall that a function $r_0 \in R^n_m[a, b]$ is said to be normal, or nondegenerate, if given any representation $r_0 = p/q$, either $\deg p = n$ or $\deg q = m$, where $\deg(\cdot)$ is the degree of a polynomial. The collection of all normal, rational functions (drawn from $R^n_m[a, b]$) we denote by $\mathcal{N}^n_m[a, b]$ or just $\mathcal{N}[a, b]$ when the context is clear. If r_0 is not normal we define the defect of r_0 by

$$\operatorname{def} r_0 = \min\{n - \deg p, m - \deg q\}.$$

By A we mean the natural parametrization of the family $R^n_m[a, b]$, $A\colon \mathbb{R}^{n+m+1} \mapsto \mathscr{A}[a, b]$, where $\mathscr{A}[a, b]$ denotes the real analytic functions over the interval $[a, b]$. If $A(\mathbf{x}_0) = r_0$ is normal, then it is well known that the map A is a local homeomorphism of a neighborhood of $\mathbf{x}_0 \in \mathbb{R}^{n+m+1}$ with a neighborhood of $r_0 \in R^n_m[a, b]$ irrespective of which L^p norm we impose on $R^n_m[a, b]$.

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The natural parametrization A has the following properties:

(i) For any $\mathbf{x}, \mathbf{y} \in S$, $A(\mathbf{x}) - A(\mathbf{y})$ has at most $n+m$ zeros (on any interval whatsoever).

(ii) If a sequence of parameters $\mathbf{x}_l \to \mathbf{x}_0$, then $(d^s/dt^s)\, A(\mathbf{x}_l) \to (d^s/dt^s)\, A(\mathbf{x}_0)$ uniformly on $[a, b]$ for $s = 0, 1, \ldots$.

The following lemma is instrumental in showing that each normal element of the rational family R^n_m is a best approximation to some continuous function outside the family.

LEMMA 1. *Suppose $f \in C^1[a, b]$, $A\colon S \subset \mathbb{R}^N \mapsto \mathscr{A}[a, b]$, and $A(\mathbf{x}_0)(t) - f(t)$ has simple zeros at a, b and K many other points, say $\{t_j\colon j = 1, \ldots, K\}$, in the interior of $[a, b]$, where A has properties* (i) *and* (ii) *as above. For each $\mathbf{x} \in S$ define $F(\mathbf{x})$ by*

$$F(\mathbf{x}) = \int_a^b |f(t) - A(\mathbf{x})(t)|\, dt.$$

Then, for any $\mathbf{h}$ in $\mathbb{R}^N$,

$$F''_+(\mathbf{x}, \mathbf{h}, \mathbf{h}) = \lim_{\lambda \to 0^+} \frac{F'(\mathbf{x} + \lambda \mathbf{h}, \mathbf{h}) - F'(\mathbf{x}, \mathbf{h})}{\lambda}$$

and

$$F''_-(\mathbf{x}, \mathbf{h}, \mathbf{h}) = \lim_{\lambda \to 0^-} \frac{F'(\mathbf{x} + \lambda \mathbf{h}, \mathbf{h}) - F'(\mathbf{x}, \mathbf{h})}{\lambda},$$

both exist and have the form

$$2 \sum_{j=1}^{K} \frac{[A'(\mathbf{x}, \mathbf{h})(t_j)]^2}{|(d\varepsilon/dt)(t_j)|} + \int_a^b sgn\varepsilon(t)\, A''(\mathbf{x}, \mathbf{h}, \mathbf{h})(t)\, dt + P,$$

where $\varepsilon(t) = f(t) - A(\mathbf{x})(t)$ and $F(x) = \int_a^b |A(\mathbf{x})(t) - f(t)|\, dt$, $\mathscr{A}[a, b]$ denotes the collection of real analytic functions on $[a, b]$, and $P = [A'(\mathbf{x}, \mathbf{h})(\omega)]^2/|(d\varepsilon/dt)(\omega)|$ or zero with $\omega = a$ or b.

Proof. The proof is exactly as the proof of Theorem 1 in Wolfe [10] save for the consideration of the endpoints of the interval. The behavior of the derivative quotient at the interior points gives rise to the terms

$$2 \sum_{j=1}^{K} \frac{[A'(\mathbf{x}, \mathbf{h})(t_j)]^2}{|(d\varepsilon/dt)(t_j)|}.$$

Hence we consider what happens at the endpoints. By virtue of our assumptions concerning A and f, for each λ sufficiently small, there are simple zeros, $t_0(\lambda)$ and $t_{K+1}(\lambda)$ of $\varepsilon(\lambda, t) = A(\mathbf{x} + \lambda\mathbf{h})(t) - f(t)$ which are continuously differentiable functions of λ and are, for sufficiently small λ, as close to a and b, respectively, as we desire. Moreover, neither $\varepsilon(\lambda, t)$ nor $\varepsilon(t) = \varepsilon(0, t)$ has any zeros on the intervals connecting a to $t_0(\lambda)$ or b to $t_{K+1}(\lambda)$. Here we have used the Implicit Function Theorem applied to $\varepsilon(\lambda, t)$ and the fact that f is continuously differentiable on an open interval containing $[a, b]$. In fact, by Taylor's Theorem, we have

$$t_0(\lambda) = a + \left.\frac{dt_0}{d\lambda}\right|_{\lambda=0} \lambda + o(\lambda)$$

and

$$t_{K+1}(\lambda) = a + \left.\frac{dt_{K+1}}{d\lambda}\right|_{\lambda=0} \lambda + o(\lambda)$$

for each λ sufficiently small. Application of the Implicit Function Theorem to $\varepsilon(\lambda, t)$ shows also that

$$\left.\frac{dt_0}{d\lambda}\right|_{\lambda=0} = \frac{-A'(\mathbf{x}, \mathbf{h})}{(d\varepsilon/dt)(a)}$$

and

$$\left.\frac{dt_{K+1}}{d\lambda}\right|_{\lambda=0} = \frac{-A'(\mathbf{x}, \mathbf{h})}{(d\varepsilon/dt)(a)}.$$

Therefore the order relation between a and $t_0(\lambda)$ is completely governed by the sign of λ with a similar remark applying at the right-hand endpoint. Let us consider what happens near $t = a$. In the difference quotient above which defines $F''_+(\mathbf{x}, \mathbf{h}, \mathbf{h})$ the term

$$\int_a^{t_0(\lambda)} \frac{\operatorname{sgn} \varepsilon(\lambda, t) - \operatorname{sgn} \varepsilon(t)}{\lambda} A'(\mathbf{x}, \mathbf{h})(t)\, dt$$

is involved precisely when $t_0(\lambda) > a$, which is to say precisely, provided λ is small enough, when

$$\frac{-A'(\mathbf{x}, \mathbf{h})}{(d\varepsilon/dt)(a)} > 0 \qquad (\dagger)$$

since $\lambda > 0$. If this last inequality holds, then the term

$$\lim_{\lambda \to 0^-} \int_a^{t_0(\lambda)} \frac{\operatorname{sgn} \varepsilon(\lambda, t) - \operatorname{sgn} \varepsilon(t)}{\lambda} A'(\mathbf{x}, \mathbf{h})(t)\, dt$$

arises in the calculation of $F''_+(\mathbf{x}, \mathbf{h}, \mathbf{h})$; otherwise it does not. Suppose it appears. Then $t_0(\lambda) > a$ and $\operatorname{sgn} \varepsilon(\lambda, t)$ and $\operatorname{sgn} \varepsilon(0, t)$ are constant on the interval $(a, t_0(\lambda))$. Suppose $\varepsilon(0, t) > 0$ there. Then, by the simplicity of the roots of $\varepsilon(0, t)$, $(d\varepsilon(0, t)/dt)|_{t=a} > 0$. Hence, using the second assumption concerning the parameter map A, for λ near enough to 0, $(d\varepsilon(\lambda, t)/dt)|_{t=t_0(\lambda)} > 0$ also. Hence

$$\operatorname{sgn} \varepsilon(\lambda, t) - \operatorname{sgn} \varepsilon(0, t) = -2$$

on $(a, t_0(\lambda)$. Similarly if $\varepsilon(0, t) < 0$,

$$\operatorname{sgn} \varepsilon(\lambda, t) - \operatorname{sgn} \varepsilon(0, t) = 2$$

on $(a, t_0(\lambda)$. Thus

$$\begin{aligned}
\lim_{\lambda \to 0^+} \int_a^{t_0(\lambda)} \frac{\operatorname{sgn} \varepsilon(\lambda, t) - \operatorname{sgn} \varepsilon(t)}{\lambda} A'(\mathbf{x}, \mathbf{h})(t)\, dt
&= \pm \frac{2}{\lambda} \int_a^{t_0(\lambda)} A'(\mathbf{x}, \mathbf{h})(t)\, dt \\
&= \pm 2A'(\mathbf{x}, \mathbf{h})(t) \frac{-A'(\mathbf{x}, \mathbf{h})}{(d\varepsilon/dt)(a)}.
\end{aligned}$$

Noting that the sign of ± 2 is opposite that of $(d\varepsilon/dt)_{t=a}$, we see that this is just

$$2 \frac{[A'(\mathbf{x}, \mathbf{h})(t_j)]^2}{|(d\varepsilon/dt)(a)|}.$$

If, on the other hand, the opposite inequality prevails in (†), this same term arises not in $F''_+(\mathbf{x}, \mathbf{h}, \mathbf{h})$ but rather in $F''_-(\mathbf{x}, \mathbf{h}, \mathbf{h})$ by the linearity of these expressions in $\mathbf{h}$. Similar arguments hold at the righthand endpoint. ∎

LEMMA 2. *Suppose $f \in C^1[a, b]$ and A satisfies the hypotheses of Lemma 1. Assume further that A is a local homeomorphism in a neighborhood of $\mathbf{x}_0$, that, with F and ε as in Lemma 1,*

$$F'(\mathbf{x}_0, \mathbf{h}) = \int_a^b \operatorname{sgn} \varepsilon(t)\, A'(\mathbf{x}_0, \mathbf{h})(t)\, dt = 0$$

for all $\mathbf{h} \in \mathbb{R}^{n+m+1}$ *and that there exists an* $\eta > 0$ *so that*

$$F''_{+}(\mathbf{x}_0, \mathbf{h}, \mathbf{h}) > \eta \qquad \text{and} \qquad F''_{-}(\mathbf{x}_0, \mathbf{h}, \mathbf{h}) > \eta$$

for all $\mathbf{h} \in \mathbb{R}^{n+m+1}$ *so that* $\|\mathbf{h}\| = 1$. *Then* $A(\mathbf{x}_0)$ *is a local best approximation to* f *from* $A(S)$ *in* L^1 *norm.*

Proof. The hypotheses concerning the sign of the one-sided derivatives $F''_{+}(\mathbf{x}_0, \mathbf{h}, \mathbf{h})$ and $F''_{-}(\mathbf{x}_0, \mathbf{h}, \mathbf{h})$ together with the compactness of the unit sphere imply that there exists a $\lambda_0 > 0$ so that, for all $\lambda < \lambda_0$ and all $\|\mathbf{h}\| = 1$,

$$F(\mathbf{x}_0 + \lambda \mathbf{h}) - F(\mathbf{x}_0) > 0.$$

This in conjunction with the fact that A is a local homeomorphism near $\mathbf{x}_0$ yields the result. ∎

We are now in a position to state and prove the following theorem.

THEOREM 1. *Suppose* $r_0 \in \mathcal{N}[a, b]$. *Then there exists a function* $f \in \mathcal{A}[a, b]$, $f \notin R^n_m[a, b]$, *so that* f *has* r_0 *as its best approximation from* $R^n_m[a, b]$ *in* $L^1[a, b]$ *and* f *interpolates* r_0 *at* a *and* b.

Proof. Let $a < t_1 < t_2 < \cdots < t_{n+m+1}$, b be the L^1 canonical points of the $n+m+1$-dimensional Haar space tangent to $R^n_m[a, b]$ at r_0. Let $w(t) = \prod_{j=0}^{n+m+2} t - t_j$, where $t_0 = a$ and $t_{n+m+2} = b$. Taking

$$f = r_0 + w \qquad \text{and} \qquad f_\lambda = \lambda f + (1 - \lambda) r_0,$$

r_0 is a critical point to each of the functions f_λ from $R^n_m[a, b]$ for $\lambda \in (0, 1)$. Note $f_\lambda \in \mathcal{A}[a, b]$. If we let

$$F_\lambda(\mathbf{x}) = \int_a^b |f_\lambda(t) - A(\mathbf{x})(t)|, dt$$

then, applying Lemma 1, we find that

$$F''_{\lambda,+}(\mathbf{x}, \mathbf{h}, \mathbf{h}) = \frac{2}{\lambda} \sum_{j=1}^{n+m+1} \frac{[A'(\mathbf{x}, \mathbf{h})(t_j)]^2}{|(d\varepsilon/dt)(t_j)|} + \int_a^b sgn\varepsilon(t)\, A''(\mathbf{x}, \mathbf{h}, \mathbf{h})(t), dt$$
$$+ \text{ a nonnegative term}$$

and

$$F''_{\lambda,-}(\mathbf{x}, \mathbf{h}, \mathbf{h}) = \frac{2}{\lambda} \sum_{j=1}^{n+m+1} \frac{[A'(\mathbf{x}, \mathbf{h})(t_j)]^2}{|(d\varepsilon/dt)(t_j)|} + \int_a^b sgn\varepsilon(t)\, A''(\mathbf{x}, \mathbf{h}, \mathbf{h})(t), dt$$
$$+ \text{ a nonnegative term,}$$

where $\mathbf{h} \in \mathbb{R}^{n+m+1}$, $\varepsilon(t) = f(t) - r_0(t)$, and the nonnegative terms are as discussed in Lemma 1. Using the compactness of the unit sphere in $\mathbb{R}^{n+m+1}$ and the continuity of

$$\mathbf{h} \mapsto A''(\mathbf{x}, \mathbf{h}, \mathbf{h})$$

we may select $\eta > 0$ so that

$$F''_{\lambda,+}(\mathbf{x}, \mathbf{h}, \mathbf{h}) > \eta \qquad \text{and} \qquad F''_{\lambda,-}(\mathbf{x}, \mathbf{h}, \mathbf{h}) > \eta$$

for all suitably small λ. Applying Lemma 2, we see that f_λ eventually has r_0 as a local best from $R^n_m[a, b]$. Moving further down the ray connecting f to r_0 we obtain f_λ for which r_0 is a global best approximation from $R^n_m[a, b]$. ∎

To extend this result to the case of defect one functions will require the following lemma.

Lemma 3. *Suppose* $r_0 \in R^n_m[a, b]$ *and* $\operatorname{def}(r_0) = 1$. *Then, given any* $\gamma \in [a, b]$, *there is a neighborhood* U *of* r_0 *in* $L^1[a, b] \cap R^n_m[a, b]$ *and a real number* $d > 0$, *so that if* $r \in U$ *the poles of* r *are further than* d *from* (α, β).

Proof. Suppose the lemma fails. Then there is a sequence, $r_j = p_j/q_j$, which has poles at β_j so that $\beta_j \to \gamma \in (-1, 1)$ and $r_j \to r_0$. Since γ is interior to $[-1, 1]$ and r_j cannot have any poles on the interval $[-1, 1]$, we may assume that each β_j is a non-real complex number. Since q_j has real coefficients and no zeros on $[-1, 1]$ it must be the case that both β_j and $\bar{\beta}_j$ are roots of q_j hence that $q_j(t) = \tilde{q}_j(t)(t^2 - |\beta_j|^2)$, where $\tilde{q}_j(t)$ is a polynomial with real coefficients of degree at most $m - 2$. Since r_j converges to r_0, it is bounded in norm. Hence we must be able to find distinct complex roots α_j and $\overline{\alpha_j}$ for p_j so that $\alpha_j \to \gamma$. So we may write $p_j(t) = \tilde{p}_j(t)(t^2 - |\alpha_j|^2)$, where $\tilde{p}_j(t)$ is a real polynomial of degree at most $n - 2$. Normalizing q_j so that $\|q_j\|_\infty = 1$ and passing if necessary to a subsequence, we may assume that p_j and q_j converge uniformly to p_0 and q_0, respectively, where $r_0 = p_0/q_0([2])$. Hence we have

$$\begin{aligned} \frac{p_0(t)}{q_0(t)} &= \frac{p^*(t)(t^2 - \gamma^2)}{q^*(t)(t^2 - \gamma^2)} \\ &= \frac{p^*(t)}{q^*(t)} \in R^{n-2}_{m-2}[-1, 1], \end{aligned}$$

where $\tilde{p}_j(t) \to p^*(t)$ and $\tilde{q}_j(t) \to q^*(t)$ uniformly on $[-1, 1]$. This contradicts our assumption concerning the defect of r_0 and so no such sequence of poles may exist.

With this lemma available we may now establish the following result:

THEOREM 2. *Let $r_0 \in R^{n-1}_{m-1}[-1, 1]$. Then there is a function $f \in C[-1, 1] \backslash R^n_m[-1, 1]$ so that r_0 is a best L^1 approximation to f from $R^n_m[-1, 1]$.*

Proof. For each $\delta \in (0, 1)$ by Theorem 2.1 we may find a function $\tilde{f}_\delta$ so that r_0 is a best L^1 approximation of $\tilde{f}_\delta$ from $R^{n-1}_{m-1}[-\delta, \delta]$ and so that $\tilde{f}_\delta(-\delta) = r_0(-\delta)$ and $\tilde{f}_\delta(\delta) = r_0(\delta)$. Define f_δ on $[-1, 1]$ by

$$f_\delta(t) = \begin{cases} \tilde{f}_\delta(t), & \text{if } t \in [-\delta, \delta]; \\ r_0(t), & \text{if } t \notin [-\delta, \delta]. \end{cases}$$

Then f_δ is a continuous function on $[-1, 1]$ and, for each δ, r_0 is a best L^1 approximation to f_δ from $R^{n-1}_{m-1}[-1, 1]$ since r_0 is best on $[-\delta, \delta]$ and the two agree on the rest of $[-1, 1]$. Suppose now that for no $\delta \in (0, 1)$ is r_0 a best L^1 approximation to f_δ from $R^n_m[-1, 1]$. Denote by f_k the function $f_{1/k}$ corresponding to the interval $[-1/k, 1/k]$. We observe that in the construction of f_k we may take the uniform norm of $f_k - r_0$ to be bounded so that $\|f_k - r_0\|_{1,[-1,1]} \to 0$ as $k \to \infty$. Hence, if r_0 is not best to any of the f_k, there exists a sequence $r_k \in \mathcal{N}^n_m[a, b]$ converging in $L^1[a, b]$ to r_0 so that

$$\int_{-1}^{1} |f_k(t) - r_k(t)|\, dt < \int_{-1}^{1} |f_k(t) - r_0(t)|\, dt \qquad \text{for each } k,$$

which is equivalent to

$$\int_{-1}^{1} |r_k(t) - r_0(t)|\, dt + \int_{-1/k}^{1/k} |f_k(t) - r_k(t)|\, dt$$
$$< \int_{-1/k}^{1/k} |r_0(t) - f_k(t)|\, dt + \int_{-1/k}^{1/k} |r_0(t) - r_k(t)|\, dt$$

or

$$\int_{-1}^{1} |r_k(t) - r_0(t)|\, dt$$
$$< \int_{-1/k}^{1/k} (|r_0(t) - f_k(t)| - |f_k(t) - r_k(t)|) + \int_{-1/k}^{1/k} |r_0(t) - r_k(t)|\, dt.$$

This in turn implies that

$$\int_{-1}^{1} |r_k(t) - r_0(t)|\, dt < 2 \int_{-1/k}^{1/k} |r_0(t) - r_k(t)|\, dt$$

by the triangle inequality, hence that

$$\frac{1}{2}<\int_{-1/k}^{1/k} \frac{|r_0(t)-r_k(t)|}{\|r_k-r_0\|_1}\,dt,$$

where the L^1 norm refers to the interval $[-1, 1]$.

Now let

$$\sigma_k=\frac{|r_0(t)-r_k(t)|}{\|r_k-r_0\|_1}.$$

We observe that the poles of the σ_k are precisely those of the r_k together with those of r_0. All of these together are bounded below in distance from any interior point of $[-1, 1]$, in particular from 0, by Lemma 3. Indeed, by a minor extension, they are bounded away from any interval $[\alpha, \beta]$, which is interior to $[-1, 1]$ from which it follows that we may extract a subsequence of σ_k, which converges uniformly to a rational function σ^* on any such interval, in particular on $[-1/2, 1/2]$, which eventually contains $[-1/k, 1/k]$. Denoting this subsequence σ_j we then have

$$\frac{2}{k_j}|\sigma_j(\xi_j)|>\frac{1}{2}$$

for each $j=1, 2, \ldots$, where $\xi_j \in [-(1/k_j), (1/k_j)]$. Since $\xi_j \to 0$ and $\sigma_j \to \sigma^*$ uniformly on every sufficiently small neighborhood of 0, we must have that σ^* has a pole at 0. But this is clearly impossible since any pole of σ^* must be a limit of poles of the σ_j all of which are bounded away from 0. Hence such a sequence of r_k cannot exist and the proof is complete. ∎

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